

Smoothing nilpotent actions on 1-manifolds

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April 1, 2014

Abstract

Let M be a connected 1-manifold, i.e., $M = \mathbb{R} \cong (0, 1), [0, 1), [0, 1]$, or S^1 , and let $\text{Homeo}_+(M)$ (resp. $\text{Diff}_+^1(M)$) be the group of orientation-preserving homeomorphisms (resp. C^1 diffeomorphisms) of M . It is a classical result that if N is a finitely-generated, torsion-free nilpotent group, then there exist 1-1 homomorphisms $\phi: N \rightarrow \text{Homeo}_+(M)$. Farb and Franks [8] show that, in fact, there exists a 1-1 homomorphism $N \rightarrow \text{Diff}_+^1(M)$. In this paper we obtain a stronger result: *every* action $\phi: N \rightarrow \text{Homeo}_+(M)$ is topologically conjugate to an action $\tilde{\phi}: N \rightarrow \text{Diff}_+^1(M)$.

1 Introduction

For convenience, we will always assume homeomorphisms and diffeomorphisms are orientation-preserving.

Given a manifold M and a homeomorphism $f: M \rightarrow M$ which is not a diffeomorphism, it is natural to ask whether f can be topologically conjugated to a (say C^1) diffeomorphism of M – that is, whether there is a homeomorphism $\psi: M \rightarrow M$ such that $\psi f \psi^{-1} \in \text{Diff}^1(M)$. If so, then (M, f) and $(M, \psi f \psi^{-1})$ are identical as topological dynamical systems; there is nothing on the level of topology that could tell us f is not differentiable. If not, (M, f) is different from any differentiable dynamical system, even on a topological level. For example:

Example 1.1. Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $f(0, 0) = (0, 0)$, and $f(r, \theta) = (r, \theta + 1/r)$ when $r \neq 0$. Note that this is a homeomorphism of the plane, but it cannot possibly be topologically conjugate to a diffeomorphism. After a conjugacy there would still be a fixed point and a foliation by invariant circles around that fixed point, which are rotated faster and faster as we approach the fixed point, which a diffeomorphism cannot do.

No such example can exist in dimension 1. On the line \mathbb{R} , there is no distinction between continuous dynamics and C^∞ dynamics. Indeed, a homeomorphism of the line is characterized up to topological conjugacy by the homeomorphism class of its fixed point set and the direction points move in the complement of its fixed point set. It is not hard to construct a C^∞ diffeomorphism $f: \mathbb{R} \rightarrow \mathbb{R}$ such that the fixed point set $\text{Fix}(f)$

is any closed set we desire, and with the direction moved on the intervals of $\mathbb{R} \setminus \text{Fix}(f)$ as desired. One simply needs to use gentle enough “bump functions” in the intervals of $\mathbb{R} \setminus \text{Fix}(f)$. (Obviously, real-analytic diffeomorphisms of the line are much more restricted; their fixed point sets cannot have any accumulation points.) For $M = [0, 1)$ or $[0, 1]$, similarly, any homeomorphism is conjugate to a C^∞ diffeomorphism.

On the circle, the situation is much more subtle, due to the possibility of complicated recurrent behavior. It is true, but not obvious, that any homeomorphism of S^1 is topologically conjugate to a C^1 diffeomorphism. This will follow, for instance, from what we do in this paper (and may already have been known). It is *not* true that a homeomorphism (or equivalently, C^1 diffeomorphism) of the circle is always topologically conjugate to a C^2 diffeomorphism, by the following theorem:

Theorem 1.2 (Denjoy [7]). *If $f: S^1 \rightarrow S^1$ is a C^2 diffeomorphism with irrational rotation number θ , then f is topologically conjugate to the rotation R_θ .*

In fact, Denjoy proved this theorem under the weaker assumption that f is C^{1+bv} . Thus, C^{1+bv} dynamics on the circle is quite different from C^1 dynamics.

To summarize what we have said so far, any homeomorphism of a 1-manifold is topologically conjugate to a C^1 diffeomorphism. In light of this, one could ask, for a 1-manifold M , which subgroups of $\text{Homeo}_+(M)$ are topologically conjugate to subgroups of $\text{Diff}_+^1(M)$? That is, which families of homeomorphisms can be simultaneously conjugated to C^1 diffeomorphisms? In this paper, we answer the question for finitely-generated virtually nilpotent subgroups:

Theorem 1.3. *Let M be a 1-manifold, i.e., $M = \mathbb{R}, [0, 1), [0, 1]$, or S^1 . Let $G \subset \text{Homeo}_+(M)$ be a finitely-generated, virtually nilpotent subgroup. Then there exists a homeomorphism $\psi: M \rightarrow M$ such that $\psi N \psi^{-1} \subset \text{Diff}_+^1(M)$.*

Thus there is no difference between C^0 and C^1 (finitely-generated, virtually) “nilpotent dynamics” in dimension 1.

Corollary 1.4. *Let $G \subset \text{Homeo}_+(S^1)$ be finitely-generated and virtually nilpotent. Given any fixed generating set for G , it is possible to conjugate G to a subgroup of $\text{Diff}_+^1(S^1)$ such that the generators are as C^1 close to rotations as desired. The same is true for $G \subset \text{Homeo}_+(\mathbb{R})$, with “rotations” replaced by “the identity.”*

This gives another method for proving Theorem B from [15]. Some remarks are in order.

Remark 1.5. Subgroups of $\text{Homeo}_+(M)$ cannot have torsion, unless $M = S^1$, and even in that case the possibilities for torsion are quite limited. Likewise, for subgroups of $\text{Homeo}_+(M)$, virtual nilpotence is not a great generalization of nilpotence. Therefore, the main content of Theorem 1.3 is to strengthen the following theorem of Farb and Franks [8].

Theorem 1.6. *Let M be a 1-manifold. Every finitely-generated, torsion-free nilpotent group is isomorphic to a subgroup of $\text{Diff}_+^1(M)$.*

Remark 1.7. Even if $M = \mathbb{R}$, C^1 cannot be improved to C^2 in the statement of Theorem 1.3. To see this, let us exhibit commuting homeomorphisms of the line which cannot simultaneously be conjugated to C^2 diffeomorphisms. Let $\bar{f}: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ be a homeomorphism with irrational rotation number having a wandering interval. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a *lift* of \bar{f} ; that is, $\pi \circ f = \bar{f} \circ \pi$, where $\pi: \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$ is the canonical projection. Then f commutes with the homeomorphism $T_1: \mathbb{R} \rightarrow \mathbb{R}, T_1(x) = x + 1$. Suppose there is a conjugacy $\psi: \mathbb{R} \rightarrow \mathbb{R}$ such that $\psi T_1 \psi^{-1}$ and $\psi f \psi^{-1}$ are (commuting) C^2 diffeomorphisms. If we quotient \mathbb{R} by $\psi T_1 \psi^{-1}$, the result is $\mathbb{R}/\psi T_1 \psi^{-1} = S^1$ with a natural C^2 structure, since $\psi T_1 \psi^{-1}$ is C^2 . (We can select a C^∞ structure inside this C^2 structure if we like.) Moreover, $\overline{\psi f \psi^{-1}}: S^1 \rightarrow S^1$ is C^2 with respect to this structure, since $\psi f \psi^{-1}$ is C^2 and commutes with $\psi T_1 \psi^{-1}$. Note that ψ induces a homeomorphism $\bar{\psi}: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\psi T_1 \psi^{-1}$, and $\bar{\psi}$ conjugates \bar{f} to $\overline{\psi f \psi^{-1}}$. But this is a contradiction; \bar{f} is not conjugate to a C^2 diffeomorphism of the circle by Denjoy's theorem, since it has a wandering interval. Therefore, the copy of \mathbb{Z}^2 generated by T_1 and f is not topologically conjugate to a subgroup of $\text{Diff}_+^2(\mathbb{R})$.

Remark 1.8. Our theorem cannot be strengthened to include the class of solvable groups. For instance, consider the following action of the Baumslag-Solitar group $BS(1, 2)$ on S^1 . Let R_θ be rotation by the irrational angle θ . Take a single R_θ orbit $\{R_\theta^n(x): n \in \mathbb{Z}\}$ and replace its points with closed intervals I_n . This gives us a larger circle, \tilde{S}^1 , and a natural surjection $\pi: \tilde{S}^1 \rightarrow S^1$ given by collapsing the intervals to points. We will define $f, g \in \text{Homeo}_+(\tilde{S}^1)$ such that $\pi f = R_\theta \pi$ and $\pi g = \pi$. Define $g_n: I_n \rightarrow I_n$ such that g_n has no fixed points in the interior of I_n and every point in $\text{Int}(I_n)$ moves clockwise for all $n \in \mathbb{Z}$. Let $f: I_n \rightarrow I_{n+1}$ be such that $f g_n = g_{n+1}^2 f$ for all $n \in \mathbb{Z}$. Since $f g f^{-1} = g^2$, $G \cong BS(1, 2)$.

It follows from Cantwell-Conlon [4] and Guelman-Liousse [10] that such an action cannot be conjugate to C^1 ; a subgroup of $\text{Diff}_+^1(S^1)$ isomorphic to $BS(1, n)$ must have a finite orbit. Bonatti, Monteverde, Navas, and Rivas [3] construct a large family of solvable subgroups of $\text{Homeo}_+(S^1)$ not conjugate to subgroups of $\text{Diff}_+^1(S^1)$.

Remark 1.9. The finite generation hypothesis in Theorem 1.3 is also necessary. For example, consider the following. Let $\phi_0: \mathbb{R} \rightarrow (-1/2, 1/2)$ be given by $\phi_n(x) = \frac{1}{\pi} \arctan(x) + n$. Now define the following homeomorphisms of \mathbb{R} : $f_1(x) = x + 1$, and for $i > 1$,

$$f_i(x) = \begin{cases} \phi_n f_{i-1} \phi_n^{-1}(x), & x \in (n - 1/2, n + 1/2) \\ x, & x \in \mathbb{Z} + 1/2 \end{cases}$$

Let H be the group generated by the f_i . Clearly, H is abelian. Note that the maximum distance a point moves under f_i goes to 0 as $i \rightarrow \infty$, since ϕ_n has derivative at most $\frac{1}{\pi}$. Using this, the reader can check that an infinite “word” of the form $\dots \circ f_2^{n_2} \circ f_1^{n_1}$ will yield a homeomorphism of the line, regardless of how the n_i are chosen. Thus we can enlarge H to the uncountable abelian group G consisting of such homeomorphisms. Note that the homeomorphisms in G are 1-periodic.

If G were conjugate to a subgroup of $\text{Diff}_+^1(\mathbb{R})$, it would be possible to make the generators f_i “gentle” enough that for any choice of n_i , the map $\dots \circ f_2^{n_2} \circ f_1^{n_1}$ is C^1 . We claim this is not the case. No matter what the generators f_i are after conjugacy, if we make the n_i increase fast enough in i , the map $\dots \circ f_2^{n_2} \circ f_1^{n_1}$ will not be Lipschitz continuous, so it cannot be C^1 .

We are not sure if the main result holds for countable virtually nilpotent subgroups of $\text{Homeo}_+(M)$.

Castro, Jorquera, and Navas [5] have shown that any finitely-generated torsion-free nilpotent group admits faithful $C^{1+\alpha}$ (differentiable with derivative α -Hölder continuous) actions on any 1-manifold for $\alpha > 0$ small enough. Indeed, let N_n be the group of $n \times n$ upper-triangular matrices with 1s on the diagonal. Farb and Franks [8] exhibit a C^1 action of this group on $[0, 1]$; Castro, Jorquera, and Navas show that this action can be smoothed to any differentiability class less than $C^{1+\frac{2}{(n-1)(n-2)}}$. (They also showed that for this action $C^{1+\alpha}$ is not possible when $\alpha > \frac{2}{(n-1)(n-2)}$, and subsequently Navas [14] showed $C^{1+\frac{2}{(n-1)(n-2)}}$ is not possible.) Since every finitely-generated torsion-free nilpotent group N embeds in some N_n , this shows N acts $C^{1+\alpha}$ for small enough α . This leads Navas to ask the natural question, what is the best regularity that can be obtained for faithful actions of N on $[0, 1]$?

Along similar lines, we ask the following question:

Question 1.10. *Is it the case that for any finitely-generated virtually nilpotent group G , and 1-manifold M , there exists $\alpha > 0$ such that for any homomorphism $\phi: G \rightarrow \text{Homeo}_+(M)$, $\phi(G)$ is conjugate to a subgroup of $\text{Diff}_+^{1+\alpha}(M)$?*

Recall the following theorem of Gromov [9]:

Theorem 1.11. *Let G be a finitely-generated group. G has polynomial growth if and only if G is virtually nilpotent.*

As a corollary, if $G \subset \text{Homeo}_+(M)$ has polynomial growth, then G is conjugate to a subgroup of $\text{Diff}_+^1(M)$. This leads us to the following question:

Question 1.12. *Suppose M is a 1-manifold, and $G \subset \text{Homeo}_+(M)$ is finitely-generated, and has sub-exponential growth. Is G conjugate to a subgroup of $\text{Diff}_+^1(M)$?*

We would like to thank Andrés Navas for bringing to our attention the following conjecture of Cantwell and Conlon [4], which relates to our question:

Conjecture 1.13. *If \mathcal{F} is a transversely orientable foliation of codimension one on a compact manifold M such that the leaves are all at finite depths and if all junctures have quasi-polynomial growth, then (M, \mathcal{F}) is homeomorphic to a C^1 -foliated manifold.*

Navas [13] has shown that there exist finitely-generated groups of intermediate (between polynomial and exponential) growth in $\text{Diff}_+^1([0, 1])$, but *not* in $\text{Diff}_+^{1+\alpha}([0, 1])$. Bleak, Kassabov, and Matucci [2] have given a structure theorem for subgroups of

$\text{Homeo}_+(S^1)$ having no non-abelian free subgroups, which may be useful in answering these questions.

Acknowledgements. I would like to thank Andrés Navas for referring me to several important articles, including the one of Cantwell-Conlon [4] and the one of Guelman-Liousse [10], for pointing out Corollary 1.4 and its relation to his article [15], and for his enthusiasm for this research.

2 Proof of Main Result

The structure of the argument is as follows. First, we recall that if G is a finitely-generated virtually nilpotent subgroup of $\text{Homeo}_+(M)$, $M = S^1$ or \mathbb{R} , then $[G, G]$ must have a global fixed point. This is an ingredient in a structure theorem (Theorem 2.4) we derive for finitely-generated virtually nilpotent subgroups of $\text{Homeo}_+(M)$, M a 1-manifold, which says that up to a nowhere dense closed set, the manifold can be split into intervals I such that for every $g \in G$, $g(I) = I$ or $g(I) \cap I = \emptyset$, and if $G_I = \{g|_I : g \in G, g(I) = I\}$, G_I is either trivial or abelian and in the latter case acts minimally on I . We then use a technique found in [8], [12] and due partly to a suggestion of Yoccoz to get functions between these intervals having nice differentiability properties. Finally, we define the lengths of the intervals carefully (in a way that depends on the *growth rate* of G) to force the action of G to be C^1 .

The following proposition is easy. See e.g. [12].

Proposition 2.1. *If $G \subset \text{Homeo}_+(S^1)$ is any amenable subgroup (not necessarily finitely-generated), there a G -invariant Borel probability measure μ . Therefore, rotation number is a homomorphism on G , so each element of $[G, G]$ has a fixed point, and for any $x \in \text{supp}(\mu)$, $x \in \bigcap_{g \in [G, G]} \text{Fix}(g)$ – in particular, $[G, G]$ has a global fixed point.*

Define a *Radon measure* on \mathbb{R} to be a Borel measure which is finite on compact subsets (note that the other usual requirements for a Radon measure are automatically satisfied on \mathbb{R}). The following result is due to Plante [17]; see also [12], Theorem 2.2.39.

Theorem 2.2. *If G is a finitely-generated virtually nilpotent subgroup of $\text{Homeo}_+(\mathbb{R})$, there is a G -invariant Radon measure on the line. Therefore, translation number is a homomorphism on G , so each element of $[G, G]$ has a fixed point, and for any $x \in \text{supp}(\mu)$, $x \in \bigcap_{g \in [G, G]} \text{Fix}(g)$ – in particular, $[G, G]$ has a global fixed point.*

It is an easy fact that for any M and $G \subset \text{Homeo}(M)$, $\text{Fix}([G, G])$ is a G -invariant set. We can see this as follows. Let $x \in \text{Fix}[G, G]$ and $g \in G$; we want to show that for any $h \in [G, G]$, $g(x) \in \text{Fix}(h)$. But $[G, G]$ is a normal subgroup of G , so $g^{-1}hg \in [G, G]$; thus, $g^{-1}hg(x) = x$, so $hg(x) = g(x)$, as desired.

Remark 2.3. Suppose $G \subset \text{Homeo}_+(\mathbb{R})$. Then it is easy to see that solvability of G would not be a sufficient assumption for Theorem 2.2 to hold. For example, consider

the group of diffeomorphisms of the line generated by T_1 and $g(x) = 2x$. This group is solvable; its commutator subgroup is the group of translations $\{T_{a/2^n} : a, n \in \mathbb{Z}\}$, which is abelian. Obviously, $[G, G]$ does not have a global fixed point.

It is also not possible to remove the assumption that the group is finitely generated. One may construct an abelian subgroup $G \subset \text{Homeo}_+(\mathbb{R})$ generated by f_1, f_2, \dots , such that $\text{Fix}(f_1) \supsetneq \text{Fix}(f_2) \supsetneq \dots$, and $\bigcap_{i=1}^{\infty} \text{Fix}(f_i) = \emptyset$; say that between two neighboring fixed points x and x' of f_{i+1} , f_{i+1} acts as a translation to the right, and between x and x' f_i has countably many fixed points comprising a single f_{i+1} -orbit. Thus every element of G has a fixed point, but G has no global fixed point.

From now on, we deal with a general 1-manifold M , which can be $\mathbb{R}, [0, 1), [0, 1]$, or S^1 .

Theorem 2.4. *Let $G \subset \text{Homeo}_+(M)$ be a finitely-generated virtually nilpotent group. There exist (countably many) open sets I_i and M_i such that the following hold:*

- $I_i \cap I_j = M_i \cap M_j = \emptyset$ unless $i = j$, and $I_i \cap M_j = \emptyset$ for all i, j .
- Each I_i and M_i is G -invariant.
- Let $I_{i,j}$ be the open intervals comprising I_i . For any j, j' , there exists $g \in G$ such that $g(I_{i,j}) = I_{i,j'}$. If $g(I_{i,j}) = I_{i,j}$, then $g|_{I_{i,j}} = \text{id}|_{I_{i,j}}$.
- For any i , the action of G on M_i is minimal. If $M = S^1$ and the action of G on S^1 is minimal, G is abelian. If some M_i is composed of open intervals $M_{i,j}$ then the group $G_{i,j} = \{g|_{M_{i,j}} : g \in G, g(M_{i,j}) = M_{i,j}\}$ is abelian.
- $\bigcup_i I_i \cup \bigcup_j M_j \subset M$ is dense.

Before proving the theorem, we need the following notions. Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be monotone increasing. We will write $f \lesssim g$ if there exists a constant $0 < C \in \mathbb{N}$ such that $f(n) \leq Cg(Cn)$ for all $n \in \mathbb{N}$. We will write $f \sim g$ if $f \lesssim g$ and $g \lesssim f$. This defines an equivalence relation. If $[f]$ and $[g]$ are equivalence classes, we may write $[f] \leq [g]$ or $[f] = [g]$ if $f \lesssim g$ or $f \sim g$, respectively. By abuse of notation, $[f] = g$ will mean $[f] = [g]$.

For the moment, let G be any finitely-generated group. Let $S = \{g_1, \dots, g_k\}$ be a symmetric finite generating set for G . This enables us to define the *norm* $|g|_S$ for $g \in G$ as

$$|g|_S = \min\{n \geq 0 : \text{there exist } g_{i_1}, \dots, g_{i_n} \in S \text{ such that } g = g_{i_1} \cdots g_{i_n}\}.$$

We define the *growth function* of G with respect to the generating set S to be

$$\mathcal{G}_{(G,S)}(n) = \#\{g \in G : |g|_S \leq n\}.$$

Though this function depends on the choice of generating set, if S' is any other choice we will have $\mathcal{G}_{(G,S)}(n) \sim \mathcal{G}_{(G,S')}(n)$, so we can talk unambiguously about the equivalence class of these, which we will denote $\mathcal{G}_G(n)$.

It is immediate that if H is a quotient of G , $\mathcal{G}_H(n) \leq \mathcal{G}_G(n)$. It is also easy to see that if $H \subset G$ is a finitely-generated subgroup then $\mathcal{G}_H(n) \leq \mathcal{G}_G(n)$. We say that G has *polynomial growth* if for some integer $d(G)$, $\mathcal{G}_G(n) = n^{d(G)}$.

Finitely-generated virtually nilpotent groups have polynomial growth. We have the following theorem, apparently derived independently by Guivarc'h [11], Bass [1], and others:

Theorem 2.5. *Let N be a finitely-generated nilpotent group, and N^i the i^{th} subgroup in the lower central series for N (with $N^1 = N$). Let $r_i = \text{rank}(N^i/N^{i+1})$ (the torsion-free rank). Then $d(N) = \sum_{i \geq 1} i \cdot r_i$.*

The quantity $d(N)$ is also referred to as the *homogeneous dimension* of N ; see e.g. [6]. If G is a finitely-generated virtually nilpotent group, and N is a (necessarily finitely-generated) nilpotent subgroup of finite index, then $d(G) = d(N)$.

It is a fact, not hard and proved in Bass [1], is that if G is a finitely-generated nilpotent group and $H \subset G$ a subgroup of infinite index, the $d(H) < d(G)$. Of course, the same holds if G is virtually nilpotent and $H \subset G$ infinite index. Indeed, if $N \subset G$ is a finite-index subgroup, then $N \cap H$ has finite index in H , so $d(N \cap H) = d(H)$; also, $N \cap H$ has infinite index in N , so by Bass, $d(N \cap H) < d(N) = d(G)$.

Now let G be our given subgroup of $\text{Homeo}_+(M)$. Let $A_i \subset G$ be the subgroup of G sending $I_{i,0}$ to itself. Thus if $g, h: I_{i,0} \rightarrow I_{i,j}$, then $h = ga$ for some $a \in A_i$; the set of group elements sending $I_{i,0}$ to $I_{i,j}$ is a left coset of A_i . Define

$$|I_{i,j}|_S = \min\{|g|_S : (g: I_{i,0} \rightarrow I_{i,j})\}.$$

Define $|M_{i,j}|_S$ similarly.

Proof of Theorem 2.4. Let us write \mathcal{I} for the set of I_i , and \mathcal{M} for the set of M_i . We have a bit of freedom in constructing \mathcal{I} , but no control over \mathcal{M} . Indeed, if M_i and M_j are minimal open sets for the action of N , we claim they are equal or disjoint. Suppose neither of these holds; then we can find $x \in M_i \cap M_j$, and without loss of generality $y \in M_i \setminus M_j$. A small open neighborhood of x must be contained in $M_i \cap M_j$, since $M_i \cap M_j$ is open. Since the orbit of y is dense in M_i and $x \in M_i$, the orbit of y must enter this small neighborhood of x , and hence enter $M_i \cap M_j$. But since M_j is a G -invariant set and $y \notin M_j$, the orbit of y cannot enter M_j , a contradiction.

First suppose that $M = S^1$, and that the action of G is minimal. We claim that G must be abelian. By Proposition 2.1, $\text{Fix}([G, G])$ is nonempty. Since this set is G -invariant, by minimality it is dense in S^1 ; therefore, $\text{Fix}([G, G]) = S^1$. Thus G is abelian.

If the action of G on S^1 is not minimal, let $x \in S^1$ be a point whose G -orbit is not dense. Since $\overline{\mathcal{O}_G(x)}$ is a G -invariant set, we may restrict attention to the open intervals in $S^1 \setminus \overline{\mathcal{O}_G(x)}$. If, given an arbitrary maximal open interval $I \subset S^1 \setminus \overline{\mathcal{O}_G(x)}$, we can find the desired sets I_i and M_i for the group $G_I = \{g|_I : g \in G, g(I) = I\}$, then we can certainly find these sets for the G -action on S^1 . We may therefore assume without loss of generality that the manifold on which G acts is an interval, say $M = \mathbb{R}$.

Suppose that M_i is some minimal open G -set, and $I \subset M_i$ is a maximal open interval. Again, let $G_I = \{g|_I : g \in G, g(I) = I\}$. As above, by applying Theorem 2.2, we can see that G_I is abelian.

If $\bigcup_i M_i \subset \mathbb{R}$ is dense, then the theorem is proved. Therefore, we assume that $\mathbb{R} \setminus \bigcup_i M_i$ has nonempty interior. Let $I \subset \mathbb{R} \setminus \bigcup_i M_i$ be an arbitrary maximal open interval. Let G_I be as before. If we can find a set of intervals I_i for the action of G_I on I having the requisite properties (including dense union in I), then we automatically get the desired intervals for the action of G on the G -orbit of I . Therefore, it suffices to restrict attention to I ; we may assume G acts on \mathbb{R} with no open minimal sets.

We will show, by induction on $d(G)$, that for a group G of homeomorphisms of \mathbb{R} with no open minimal subsets we can find the desired permuted intervals. If $d(G) = 0$, then since $G \subset \text{Homeo}_+(\mathbb{R})$, G is torsion-free, and therefore trivial. Thus the result is trivial in this case; we let \mathcal{I} simply contain \mathbb{R} .

If $d(G) = 1$, it is not hard to see that $G \cong \mathbb{Z}$. Indeed, G contains a nilpotent subgroup of finite index N such that $d(N) = 1$, and a torsion-free nilpotent group with $d(N) = 1$ must be isomorphic to \mathbb{Z} (any larger nilpotent group would contain a copy of \mathbb{Z}^2 , and $d(\mathbb{Z}^2) = 2$). Thus G is a finite torsion-free extension of a group isomorphic to \mathbb{Z} , so G itself is isomorphic to \mathbb{Z} . Therefore, G is generated by a single element g . If $\text{Fix}(g)$ has nonempty interior, we may add the maximal open intervals in $\text{Fix}(g)$ to \mathcal{I} . Thus we may remove the fixed points of g , and show that we get the desired interval structure on the remaining open intervals I . Let x be an arbitrary element of such an I , and let I_n be the interval $(g^n(x), g^{n+1}(x))$. This gives us the desired interval structure.

Now let $d(G)$ be arbitrary, and assume the result holds for finitely-generated virtually nilpotent subgroups of $\text{Homeo}_+(\mathbb{R})$ with no open minimal regions having growth degree smaller than that of G . We may remove any global fixed points of the action of G on \mathbb{R} , putting any open intervals contained therein into \mathcal{I} . Thus we may assume that G has no global fixed points. Obviously, $(\mathbb{R} \setminus \text{Fix}([G, G])) \cup \text{Int}(\text{Fix}([G, G])) \subset \mathbb{R}$ is dense, and these two sets are G -invariant. Therefore, it suffices to find the desired intervals in $\mathbb{R} \setminus \text{Fix}([G, G])$ and in $\text{Int}(\text{Fix}([G, G]))$.

To find the necessary intervals for $\mathbb{R} \setminus \text{Fix}([G, G])$, it suffices to do this for an arbitrary maximal open interval $I \subset \mathbb{R} \setminus \text{Fix}([G, G])$. As before, form G_I by taking the subgroup of G sending I to itself, followed by the quotient of this subgroup that we get by restriction to I . We have taken a quotient of a subgroup of infinite index, since if $g(I) \neq I$ then $g^n(I) \neq I$ for all $n > 0$. Therefore, as noted above, we have that $d(G_I) < d(G)$. So by inductive hypothesis we can find the desired intervals for the action of G_I on I .

Similarly, if I is a maximal open interval in $\text{Int}(\text{Fix}([G, G]))$ then the group G_I satisfies $d(G_I) < d(G)$, completing the argument by induction, *unless* G is abelian (so I is all of \mathbb{R}). In that case we have $G \cong \mathbb{Z}^k$ for some $k > 1$. We have assumed that G has no global fixed points, so some $f \in G$ is conjugate to a translation and we can consider $\bar{G} = G/f$ to be a subgroup of $\text{Homeo}_+(S^1)$ isomorphic to \mathbb{Z}^{k-1} . \bar{G} does not act minimally on S^1 , or else G would have acted minimally on \mathbb{R} . Therefore, \bar{G} has a minimal invariant closed set $X \subset S^1$ which is either finite or a Cantor set. For I

a maximal open interval in $S^1 \setminus X$, we can take \bar{G}_I , the quotient of a subgroup of \bar{G} acting on I . We have $d(\bar{G}_I) \leq k - 1 < k$, so by inductive hypothesis we can find the desired interval structure for the action of \bar{G}_I on I and hence for G on \mathbb{R} . \square

Thus there is an open and dense set of intervals in M on which G acts either by permutation or minimally. It will be useful to us to define a family of diffeomorphisms $\phi_{\alpha,\beta}: (0, \alpha) \rightarrow (0, \beta)$, $\alpha, \beta \in \mathbb{R}_{>0}$ which is *equivariant*, i.e. such that $\phi_{\beta,\gamma} \circ \phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$. (Thus we have the structure of a groupoid, whose objects are intervals and whose morphisms are the diffeomorphisms.) Following Farb-Franks [8] and Navas [12], who credit J. C. Yoccoz for the idea, for any $\alpha > 0$ we define $\phi_\alpha: \mathbb{R} \rightarrow (-\alpha/2, \alpha/2)$ by $\phi_\alpha(x) = \frac{\alpha}{\pi} \arctan(\alpha x)$. Then we define $\phi_{\alpha,\beta}: (-\alpha/2, \alpha/2) \rightarrow (-\beta/2, \beta/2)$ by $\phi_{\alpha,\beta} = \phi_\beta \phi_\alpha^{-1}$.

It is immediate from this definition that the diffeomorphisms $\phi_{\alpha,\beta}$ are equivariant. They have very well-behaved differentiability properties, namely that $\phi'_{\alpha,\beta}(-\alpha/2) = \phi'_{\alpha,\beta}(\alpha/2) = 1$, and $\phi'_{\alpha,\beta}$ is uniformly close to 1 provided that β/α is close to 1. More precisely, Navas [12] shows that when $\alpha \geq \beta$,

$$\sup_{x \in (-\alpha/2, \alpha/2)} |1 - \phi'_{\alpha,\beta}(x)| = 1 - \frac{\beta^2}{\alpha^2}.$$

It is these properties which make this family useful to us. If $I = (x - \alpha/2, x + \alpha/2)$, $J = (y - \beta/2, y + \beta/2)$, we will abuse notation and write $\phi_{\alpha,\beta}: I \rightarrow J$ where we really mean $T_y \phi_{\alpha,\beta} T_{-x}: I \rightarrow J$. We will similarly abuse notation for intervals lying on S^1 .

Lemma 2.6. *Suppose we are given a set of finite disjoint open intervals $\{I_i \subset M: i \in \mathbb{Z}\}$ such that $U = \cup_i I_i \subset M$ is dense. Let l_i be the length of I_i for each i . Let l'_i be positive real numbers such that the following condition is satisfied: whenever $S \subset \mathbb{Z}$ is such that $\cup_{i \in S} I_i$ is contained in some bounded interval, $\sum_{i \in S} l'_i < \infty$. In particular, this holds if $\sum_{i \in \mathbb{Z}} l'_i < \infty$. Then there exists a homeomorphism $\phi: M \rightarrow \phi(M)$ such that the length of $\phi(I_i)$ is l'_i , and $m(\phi(C)) = 0$, where $C = U^c$ and m is the Lebesgue measure.*

Proof. Assume $M = \mathbb{R}$; the other cases are almost the same. Define $\phi(0) = 0$. Without loss of generality, $0 \in C$. Let $x > 0$ (the case $x < 0$ is similar). If $x \in C$, let $\phi(x) = \sum_{I_i \subset (0,x)} l'_i$. If $x \in I_j = (a_j, b_j)$, let $\phi(x) = \sum_{I_i \subset (0,x)} l'_i + (x - a_j) \frac{l'_j}{l_j}$. First, this really does define a function on the whole real line since by assumption $\sum_{I_i \subset (0,x)} l'_i < \infty$. This function ϕ is monotone: if $y > x$, then since $U \subset \mathbb{R}$ is dense there is (at least part of) an interval I_i lying between x and y . We claim it is also continuous. If x lies in an interval, continuity at x is obvious. Suppose $x \in C$, and $(x_n)_{n \geq 1} \subset C$ is a strictly increasing sequence of points approaching x from the left with $x_1 > 0$. Note that $\phi(x) = \sum_{I_i \subset (0,x)} l'_i = \lim_{n \rightarrow \infty} \sum_{I_i \subset (0,x_n)} l'_i$, since $I_i \subset (0,x)$ implies $I_i \subset (0,x_n)$ for some n . But the right-hand side is just $\lim_{n \rightarrow \infty} \phi(x_n)$. Therefore, ϕ is a monotone-increasing, continuous function, hence a homeomorphism onto its image. Notice that $m(\phi(C)) = 0$, since for $y > x$, $\phi(y) - \phi(x)$ depends only on intervals lying between x and y . \square

Recall that we write $I_{i,j}$ (resp. $M_{i,j}$) for the open intervals making up the sets I_i (resp. M_i) in \mathcal{I} (resp. \mathcal{M}). The numbering is arbitrary. After applying a topological conjugacy, we want these intervals (which by abuse of notation we will still write $I_{i,j}$ or $M_{i,j}$) to have certain lengths, which we now define. Let $d = d(G)$.

We will make the length of $I_{i,j}$ after conjugacy

$$\ell_{i,j} = \frac{1}{(2^{\frac{|i|}{d+1}} + |I_{i,j}|_S)^{d+1}},$$

and in exactly the same way we will make the length of $M_{i,j}$ after conjugacy

$$\ell'_{i,j} = \frac{1}{(2^{\frac{|i|}{d+1}} + |M_{i,j}|_S)^{d+1}}.$$

Proposition 2.7. *The sum $\sum_{i,j} \ell_{i,j} + \ell'_{i,j} < \infty$.*

Proof. Let us consider $\sum_{i,j} \ell_{i,j}$, the other sum being similar. Fix i . Note that $\#\{g \in G : |g|_S = n\} \sim n^{d-1}$. Therefore, $\#\{j : |I_{i,j}|_S = n\} \lesssim n^{d-1}$, since $|I_{i,j}|_S = n$ implies that there is g of length n such that $g : I_{i,0} \rightarrow I_{i,j}$, but there may be more than one such g . Thus modulo constants, the sum $\sum_j \ell_{i,j}$ is bounded by $\sum_{n \geq 0} \frac{n^{d-1}}{(2^{\frac{|i|}{d+1}} + n)^{d+1}}$. This sum clearly converges, since it has the form $\sum 1/n^2$. In fact, the reader can check that the value to which it converges decays exponentially as $|i| \rightarrow \infty$, so $\sum_{i,j} \ell_{i,j} < \infty$. \square

Therefore, by Lemma 2.6, there exists a conjugacy after which the intervals have these lengths and their complement has measure 0.

We apply a further topological conjugacy, after which the action of G has the following form. The unique map from $I_{i,0}$ to $I_{i,j}$ under the action of G is defined to be $\phi_{\ell_{i,0}, \ell_{i,j}}$ (so the map from $I_{i,j}$ to $I_{i,k}$ is $\phi_{\ell_{i,j}, \ell_{i,k}}$). For each j , we arbitrarily choose $g_{i,j} \in G : M_{i,0} \rightarrow M_{i,j}$ such that $|g_{i,j}|_S = |M_{i,j}|_S$ (so $g_{i,j}$ is an element of minimum length among those sending $M_{i,0}$ to $M_{i,j}$). We define $g_{i,j}$ to act via $\phi_{\ell'_{i,0}, \ell'_{i,j}}$. Finally, we must describe how G acts on $M_{i,0}$ after conjugacy. Let $G_i = G_{M_{i,0}} = \{g|_{M_{i,0}} : g \in G, g(M_{i,0}) = M_{i,0}\}$. Recall from before that G_i is abelian.

We ask that every element of G_i have the form $\phi_{\ell'_{i,0}} T_\alpha \phi_{\ell'_{i,0}}^{-1}$, $\alpha \in \mathbb{R}$. If the generators of G_i are h_1, \dots, h_k , and $h_n = \phi_{\ell'_{i,0}} T_{\alpha_n} \phi_{\ell'_{i,0}}^{-1}$ for $1 \leq n \leq k$, we will show that if we choose the α_n small enough then the generators of G will have derivative uniformly close to 1 on M_i .

Lemma 2.8. *Let $C \subset M$ be a closed set of measure 0. If $f : M \rightarrow M$ is a homeomorphism such that $f(C) = C$, and f is C^1 on $M \setminus C$ with $f'(x_n) \rightarrow 1$ for any sequence of points $x_n \in M \setminus C$ with $x_n \rightarrow x \in C$, then f is C^1 on M and has derivative 1 on C .*

Proof. Let $x \in C$. Suppose $y > x$ is very close to x . (If $M = S^1$, choose an orientation.) The set (x, y) is the union of $C \cap (x, y)$ and a countable collection of open intervals I_i . $f(I_i)$ is an interval of length approximately $\ell(I_i)$, since $f' \approx 1$ on $(x, y) \setminus C$.

Now $f((x, y)) = f(C \cap (x, y)) \cup \bigcup_i f(I_i)$; letting m be the Lebesgue measure, we have $m(\bigcup_i f(I_i)) \approx m(\bigcup_i I_i) = y - x$, since C has measure 0. Moreover, $m(f(C \cap (x, y))) = 0$ since C has measure 0 and $f(C \cap (x, y)) \subset C$. So $f(x, y)$ is an interval of measure (i.e. length) approximately $y - x$, which implies that $\frac{f(y) - f(x)}{y - x} \approx 1$, and so $f'(x) = 1$. \square

Before we finish the proof of Theorem 1.3, we need a definition. Let G be a finitely-generated group, and H a finitely-generated subgroup. Let S, T be finite generating sets of G and H , respectively. We define the *distortion* to be

$$\text{dist}(H, T; G, S)(n) = \text{diam}_T(B_S(n) \cap H),$$

where $B_S(n)$ is the ball of radius n about the identity in G measured in the word metric coming from S , and diam_T is the diameter measured in the word metric coming from T . Although this definition depends on the chosen generating sets S and T , its growth type does not; as for growth, if S' and T' are different generating sets of G and H respectively, then $\text{dist}(H, T'; G, S')(n) \sim \text{dist}(H, T; G, S)(n)$, so we can speak unambiguously about the equivalence class $\text{dist}(H, G)(n)$. For background, see e.g. [9].

If G is a finitely-generated virtually nilpotent group, and H is any subgroup, we have $\text{dist}(H, G)(n) \lesssim \mathcal{G}_G(n)$. To see this, we may pass to a finite-index torsion-free nilpotent subgroup $N \subset G$, as this will be quasi-isometric to G . Since $d(N) = \sum_{i \geq 1} i \cdot r_i$, and $r_k \geq 1$ where k is the nilpotence class of N (because N is torsion-free), we have $d(N) \geq k$. On the other hand, for any $H \subset N$, $\text{dist}(H, N)(n) \lesssim n^k$ (for an exact expression of $\text{dist}(H, N)(n)$, see [16]).

Proof of Theorem 1.3. Note that each generator g_n for G is obviously C^1 on the union of the intervals in \mathcal{I} and \mathcal{M} . By Lemma 2.8, it suffices to show that for any sequence of points $x_m \in U$ approaching a point x in the complement C , the derivatives of g_n approach 1. It suffices to consider the case in which x_m does not visit any interval infinitely many times – if x_m visits an interval infinitely many times, the subsequence of x_m lying in that interval must approach an endpoint of that interval, and clearly g'_n must approach 1 on this subsequence. The length of the interval containing x_m must go to 0 as $m \rightarrow \infty$. Therefore, it is enough to show that g_n has derivative uniformly close to 1 on small intervals.

On a (small) interval $I_{i,j}$, this is easy. Indeed, by construction $|g_n(I_{i,j})|_S$ can differ from $|I_{i,j}|_S$ by at most 1. Therefore, $|1 - \frac{\ell(g_n(I_{i,j}))}{\ell_{i,j}}|$ is $O(\ell_{i,j}^{\frac{1}{d+1}})$ as $\ell_{i,j} \rightarrow 0$, and since g_n goes from $I_{i,j}$ to $g_n(I_{i,j})$ by a member of our equivariant family $\{\phi_{\alpha,\beta}\}$, $\max_{x \in I_{i,j}} |g'_n(x) - 1|$ is also $O(\ell_{i,j}^{\frac{1}{d+1}})$.

Now consider the action of g_n on a small interval $M_{i,j}$. Say $g_n: M_{i,j} \rightarrow M_{i,j'}$. Note that, if $g_{i,j}$ and $g_{i,j'}$ are our chosen maps from $M_{i,0}$ to $M_{i,j}$ and $M_{i,j'}$ respectively, then

$$g_n = (g_{i,j'} g_{i,j}^{-1})(g_{i,j} [g_{i,j'}^{-1} g_n g_{i,j}] g_{i,j}^{-1}).$$

Here, $g_{i,j'}^{-1} g_n g_{i,j}$ sends $M_{i,0}$ to itself. It is a word of length at most $2|M_{i,j}|_S + 2$, since $|g_{i,j'}|_S \leq |M_{i,j}|_S + 1$ and $|g_n|_S = 1$. Recall that the maximum degree of distortion in G

is the nilpotence class k of a finite-index subgroup $N \subset G$, which in particular is less than $d = d(G)$. The length of $g_{i,j}^{-1}g_n g_{i,j}$ in terms of a generating set for G_i will be at most of order $|M_{i,j}|_S^k$, and hence $\phi_{\ell'_{i,0}}^{-1}g_{i,j}^{-1}g_n g_{i,j}\phi_{\ell'_{i,0}}$ will be translation by an amount of order at most $|M_{i,j}|_S^k$. When we conjugate this by $g_{i,j}$, the result is of the form $\phi_\beta T_\alpha \phi_\beta^{-1}$, where $\alpha \lesssim |M_{i,j}|_S^k$ and $\beta \sim \frac{1}{|M_{i,j}|_S^{d+1}}$.

We claim that if $\alpha\beta$ is small, then $\phi_\beta T_\alpha \phi_\beta^{-1}$ has derivative uniformly close to 1. In fact,

$$\phi_\beta T_\alpha \phi_\beta^{-1}(x) = \frac{\beta}{\pi} \arctan(\alpha\beta + \tan(\frac{\pi x}{\beta})),$$

which has derivative

$$\frac{d}{dx} \phi_\beta T_\alpha \phi_\beta^{-1} = \frac{1 + \tan^2(\frac{\pi x}{\beta})}{1 + (\alpha\beta + \tan(\frac{\pi x}{\beta}))^2},$$

which has reciprocal $1 + \frac{2\alpha\beta \tan(\frac{\pi x}{\beta}) + \alpha^2 \beta^2}{1 + \tan^2(\frac{\pi x}{\beta})}$. Obviously $\frac{\alpha^2 \beta^2}{1 + \tan^2(\frac{\pi x}{\beta})}$ is $O((\alpha\beta)^2)$ as $\alpha\beta \rightarrow 0$.

Moreover, $\frac{\tan(\frac{\pi x}{\beta})}{1 + \tan^2(\frac{\pi x}{\beta})}$ has magnitude bounded by $1/2$, so $|\frac{2\alpha\beta \tan(\frac{\pi x}{\beta})}{1 + \tan^2(\frac{\pi x}{\beta})}| \leq \alpha\beta$ for all x .

Therefore, $g_{i,j}[g_{i,j}^{-1}g_n g_{i,j}]g_{i,j}^{-1}$ has derivative close to 1 for $|M_{i,j}|_S$ large enough. If we choose the translation speeds of the generators of G_i to be small enough, then $g_{i,j}[g_{i,j}^{-1}g_n g_{i,j}]g_{i,j}^{-1}$ will have derivative close to 1 everywhere on M_i . And as we have seen, $g_{i,j}g_{i,j}^{-1}$ has derivative close to 1, since $\ell'_{i,j} \approx \ell'_{i,j'}$. This proves the theorem. \square

Proof of Corollary 1.4. The proof is immediate, by choosing the lengths of the intervals appropriately and making the minimal abelian actions on the $M_{i,j}$ close enough to the identity in the C^1 topology. \square

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